

A NEW SUPER CONGRUENCE INVOLVING MULTIPLE HARMONIC SUMS

LIUQUAN WANG

ABSTRACT. Let \mathcal{P}_n denote the set of positive integers which are prime to n . Let B_n be the n -th Bernoulli number. For any prime $p > 5$ and integer $r \geq 2$, we prove that

$$\sum_{\substack{l_1+l_2+\dots+l_5=p^r \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv -\frac{5!}{6} p^{r-1} B_{p-5} \pmod{p^r}.$$

This gives an extension of a family of super congruences found by Wang, Cai and Zhao.

1. INTRODUCTION

Zhao [6] first discovered a curious congruence

$$\sum_{\substack{i+j+k=p \\ i, j, k > 0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}. \quad (1.1)$$

Here $p \geq 3$ is a prime and B_n is the n -th Bernoulli number, which is defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n} x^n.$$

Zhou and Cai [8] generalized this congruence to the case of arbitrary number of variables. They showed that for any prime $p \geq 5$ and positive integer $n \leq p-2$,

$$\sum_{\substack{l_1+l_2+\dots+l_n=p, \\ l_1, l_2, \dots, l_n > 0}} \frac{1}{l_1 l_2 \cdots l_n} \equiv \begin{cases} -(n-1)! B_{p-n} \pmod{p}, & \text{if } 2 \nmid n; \\ -\frac{n}{2(n+1)} n! B_{p-n-1} \pmod{p^2}, & \text{if } 2 \mid n. \end{cases} \quad (1.2)$$

In another direction, Wang and Cai [4] gave a new generalization of (1.1) by replacing prime p to any prime power. Let \mathcal{P}_n denote the set of positive integers which are prime to n . They proved that for any prime $p \geq 3$,

$$\sum_{\substack{i+j+k=p^r \\ i, j, k \in \mathcal{P}_p}} \frac{1}{ijk} \equiv -2p^{r-1} B_{p-3} \pmod{p^r}. \quad (1.3)$$

Date: June 1, 2014.

2010 Mathematics Subject Classification. Primary 11A07, 11A41.

Key words and phrases. super congruences, Bernoulli numbers, harmonic sums.

Zhao [7] extended this result to the case when there are four variables. He proved that for and prime $p \geq 5$ and integer $r \geq 2$,

$$\sum_{\substack{l_1+\dots+l_4=p^r \\ l_1, \dots, l_4 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4} \equiv -\frac{4!}{5} p^r B_{p-5} \pmod{p^{r+1}}. \quad (1.4)$$

In viewing of congruences (1.3) and (1.4), for $1 \leq k \leq n$ we define

$$S_n^{(k)}(p^r) = \sum_{\substack{l_1+\dots+l_n=kp^r \\ l_i < p^r, l_i \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n}.$$

It would be very interesting to find a general congruence for $S_n^{(1)}(p^r)$ modulo p^r when n is odd or modulo p^{r+1} when n is even. The cases $n = 3$ and 4 have been solved from (1.3) and (1.4). As n increases, the problem becomes much more difficult. The main goal of this paper is to establish the result for the case $n = 5$.

Theorem 1. *Let $p > 5$ be a prime and $r \geq 2$ be an integer. We have*

$$\sum_{\substack{l_1+l_2+\dots+l_5=p^r \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv -\frac{5!}{6} p^{r-1} B_{p-5} \pmod{p^r}.$$

While $r = 1$, by (1.2) the congruence should be $S_5^{(1)}(p) \equiv -4! B_{p-5} \pmod{p}$. This is different to the case $r \geq 2$. From (1.2) and (1.4), this phenomenon also happens for the case $n = 4$.

Theorem 2. *Let $p > 5$ be a prime and n be a positive integer. Suppose $p^r | n$ but $p^{r+1} \nmid n$ for some positive integer r .*

(i) *If $r = 1$, then*

$$\sum_{\substack{l_1+l_2+\dots+l_5=n \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv -\frac{4!}{6} \cdot \left(5 \cdot \frac{n}{p} + \left(\frac{n}{p} \right)^3 \right) B_{p-5} \pmod{p}.$$

(ii) *If $r \geq 2$, then*

$$\sum_{\substack{l_1+l_2+\dots+l_5=n \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv -\frac{5!}{6} \cdot \frac{n}{p} B_{p-5} \pmod{p^r}.$$

In particular, if $n = p^r$, then part (ii) of Theorem 2 becomes Theorem 1.

Different from the method used for proving (1.3)-(1.4) (see [4, 7]), the idea to prove Theorem 1 is to establish the recurrence relation

$$S_5^{(1)}(p^{r+1}) \equiv p S_5^{(1)}(p^r) \pmod{p^{r+1}}, \quad r \geq 2.$$

And then we only need to prove that $S_5^{(1)}(p^2) \equiv -\frac{5!}{6} p B_{p-5} \pmod{p^2}$. This idea can also be applied to give a new proof of (1.3) and (1.4).

2. PRELIMINARIES

Lemma 1. *Let $p > n$ be a prime and $1 \leq k \leq n - 1$. For any integer $r \geq 1$, we have*

$$(i) S_n^{(k)}(p^r) \equiv (-1)^n S_n^{(n-k)}(p^r) \pmod{p^r}.$$

$$(ii) S_n^{(1)}(p^{r+1}) \equiv \sum_{k=1}^{n-1} \binom{p-k+n-1}{n-1} S_n^{(k)}(p^r) \pmod{p^{r+1}}.$$

Proof. (i) Since $(y_1, \dots, y_n) \leftrightarrow (p^r - y_1, \dots, p^r - y_n)$ gives a bijection between the solutions of

$$y_1 + \dots + y_n = kp^r, \quad y_i \in \mathcal{P}_p, \quad y_i < p^r, \quad 1 \leq i \leq n$$

and

$$y_1 + \dots + y_n = (n-k)p^r, \quad y_i \in \mathcal{P}_p, \quad y_i < p^r, \quad 1 \leq i \leq n$$

we have

$$\begin{aligned} S_n^{(k)}(p^r) &= \sum_{\substack{y_1 + \dots + y_n = kp^r \\ y_i < p^r, y_i \in \mathcal{P}_p}} \frac{1}{y_1 \cdots y_n} \\ &= \sum_{\substack{y_1 + \dots + y_n = (n-k)p^r \\ y_i < p^r, y_i \in \mathcal{P}_p}} \frac{1}{(p^r - y_1) \cdots (p^r - y_n)} \\ &\equiv (-1)^n S_n^{(n-k)}(p^r) \pmod{p^r}. \end{aligned}$$

(ii) For any n -tuple (l_1, \dots, l_n) of positive integers satisfying $l_1 + l_2 + \dots + l_n = p^{r+1}$ and $l_1, \dots, l_n \in \mathcal{P}_p$, we rewrite them as

$$l_i = x_i p^r + y_i, \quad 0 \leq x_i < p, \quad 1 \leq y_i < p^r, \quad y_i \in \mathcal{P}_p, \quad 1 \leq i \leq n.$$

Since $(p - \sum_{i=1}^n x_i)p^r = \sum_{i=1}^n y_i < np^r$, we know there exists $1 \leq k \leq n-1$ such that

$$\begin{cases} x_1 + x_2 + \dots + x_n = p - k \\ y_1 + y_2 + \dots + y_n = kp^r \end{cases}.$$

Hence

$$S_n^{(1)}(p^{r+1}) = \sum_{k=1}^{n-1} \sum_{\substack{x_1 + \dots + x_n = p-k \\ y_1 + \dots + y_n = kp^r \\ x_i \geq 0, y_i < p^r, y_i \in \mathcal{P}_p}} \frac{1}{(x_1 p^r + y_1) \cdots (x_n p^r + y_n)}.$$

Given $1 \leq k \leq n-1$, the equation $x_1 + x_2 + \dots + x_n = p-k$ has $\binom{p-k+n-1}{n-1}$ solutions (x_1, x_2, \dots, x_n) of nonnegative integers. Because

$$(x_1 p^r + y_1) \cdots (x_n p^r + y_n) \equiv \left(y_1 \cdots y_n \sum_{i=1}^n \frac{x_i}{y_i} \right) p^r + y_1 \cdots y_n \pmod{p^{2r}},$$

and

$$\sum_{\substack{x_1 + \dots + x_n = p-k \\ x_i \geq 0, 1 \leq i \leq n}} x_i = \frac{1}{n} \sum_{\substack{x_1 + \dots + x_n = p-k \\ x_i \geq 0, 1 \leq i \leq n}} (x_1 + \dots + x_n) = \frac{p-k}{n} \binom{p-k+n-1}{n-1} \equiv 0 \pmod{p}.$$

We have

$$\begin{aligned}
S_n^{(1)}(p^{r+1}) &\equiv \sum_{k=1}^{n-1} \sum_{\substack{x_1+\dots+x_n=p-k \\ x_i \geq 0, 1 \leq i \leq n}} \sum_{\substack{y_1+\dots+y_n=kp^r \\ y_i < p^r, y_i \in \mathcal{P}_p, 1 \leq i \leq n}} \frac{1}{(y_1 \cdots y_n \sum_{i=1}^n \frac{x_i}{y_i})p^r + y_1 \cdots y_n} \\
&\equiv \sum_{k=1}^{n-1} \sum_{\substack{x_1+\dots+x_n=p-k \\ x_i \geq 0, 1 \leq i \leq n}} \sum_{\substack{y_1+\dots+y_n=kp^r \\ y_i < p^r, y_i \in \mathcal{P}_p, 1 \leq i \leq n}} \frac{y_1 \cdots y_n - (y_1 \cdots y_n \sum_{i=1}^n \frac{x_i}{y_i})p^r}{(y_1 \cdots y_n)^2} \\
&\equiv \sum_{k=1}^{n-1} \binom{p-k+n-1}{n-1} \sum_{\substack{y_1+\dots+y_n=kp^r \\ y_i < p^r, y_i \in \mathcal{P}_p, 1 \leq i \leq n}} \frac{1}{y_1 \cdots y_n} \\
&\quad - p^r \sum_{k=1}^{n-1} \sum_{i=1}^n \sum_{\substack{x_1+\dots+x_n=p-k \\ x_j \geq 0, 1 \leq j \leq n}} x_i \sum_{\substack{y_1+\dots+y_n=kp^r \\ y_j < p^r, y_j \in \mathcal{P}_p, 1 \leq j \leq n}} \frac{1}{y_1 \cdots y_{i-1} y_i^2 y_{i+1} \cdots y_n} \\
&\equiv \sum_{k=1}^{n-1} \binom{p-k+n-1}{n-1} \sum_{\substack{y_1+\dots+y_n=kp^r \\ y_i < p^r, y_i \in \mathcal{P}_p, 1 \leq i \leq n}} \frac{1}{y_1 \cdots y_n} \pmod{p^{r+1}}.
\end{aligned}$$

This completes the proof of (i). \square

Lemma 2. Let $p > 5$ be a prime. For $1 \leq a \leq 4$, we denote by C_a the number of solutions (x_1, \dots, x_5) of nonnegative integers of the equation

$$x_1 + \cdots + x_5 = 2p - a, \quad 0 \leq x_i < p, \quad 1 \leq i \leq 5.$$

Then we have

$$(i) \ C_a = \binom{2p-a+4}{4} - 5 \binom{p-a+4}{4}.$$

$$(ii) \ C_1 \equiv -\frac{3}{4}p \pmod{p^2}, \ C_2 \equiv \frac{1}{4}p \pmod{p^2}, \ C_3 \equiv -\frac{1}{4}p \pmod{p^2}, \ C_4 \equiv \frac{3}{4}p \pmod{p^2}.$$

$$(iii) \ \sum_{\substack{x_1+\dots+x_5=2p-a \\ 0 \leq x_i < p, 1 \leq i \leq n}} x_1 \equiv 0 \pmod{p}.$$

Proof. (i) Note that for any solution (x_1, \dots, x_5) of the equation

$$x_1 + \cdots + x_5 = 2p - a, \quad x_i \geq 0, \quad 1 \leq i \leq 5,$$

at most one of x_i ($1 \leq i \leq 5$) can be greater than or equal to p .

By Inclusion-Exclusion Principle, we deduce that

$$\begin{aligned}
C_a &= \#\left\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + \cdots + x_5 = 2p - a, x_i \geq 0, 1 \leq i \leq 5\right\} \\
&\quad - 5 \sum_{k=0}^{p-a} \#\left\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + \cdots + x_4 = p - a - k, x_5 = p + k, 0 \leq x_i < p, 1 \leq i \leq 4\right\} \\
&= \binom{2p-a+4}{4} - 5 \sum_{k=0}^{p-a} \binom{p-a-k+3}{3} \\
&= \binom{2p-a+4}{4} - 5 \sum_{k=3}^{p-a+3} \binom{k}{3} \\
&= \binom{2p-a+4}{4} - 5 \binom{p-a+4}{4}.
\end{aligned}$$

(ii) For $u = 1$ or 2 we have

$$\binom{up-a+4}{4} = \frac{(up-a+4)(up-a+3)(up-a+2)(up-a+1)}{4!}.$$

Since $1 \leq a \leq 4$, we deduce that

$$\binom{up-a+4}{4} \equiv \frac{(-1)^{a-1}(a-1)!(4-a)!}{4!} up \pmod{p^2}.$$

From which the congruences in (ii) follows by (i) and some simple calculations.

(iii) By (ii) we have $C_a \equiv 0 \pmod{p}$. Hence

$$\sum_{\substack{x_1+\cdots+x_5=2p-a \\ 0 \leq x_i < p, 1 \leq i \leq 5}} x_1 = \frac{1}{5} \sum_{\substack{x_1+\cdots+x_5=2p-a \\ 0 \leq x_i < p, 1 \leq i \leq 5}} (x_1 + \cdots + x_5) = \frac{2p-a}{5} C_a \equiv 0 \pmod{p}.$$

□

Lemma 3. (Cf. [8]) Let $r, \alpha_1, \dots, \alpha_n$ be positive integers, $r = \alpha_1 + \cdots + \alpha_n \leq p-3$. Then

$$\sum_{\substack{1 \leq l_1, \dots, l_n \leq p-1 \\ l_i \neq l_j, \forall i \neq j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \cdots l_n^{\alpha_n}} \equiv \begin{cases} (-1)^n (n-1)! \frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}, & \text{if } 2 \nmid r; \\ (-1)^{n-1} (n-1)! \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}, & \text{if } 2 \mid r. \end{cases}$$

This leads to the following corollary.

Corollary 1. Let α be a positive integer and $p \geq \alpha + 3$ a prime. Then

$$\sum_{1 \leq l < p} \frac{1}{l^\alpha} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } 2 \nmid \alpha; \\ 0 \pmod{p} & \text{if } 2 \mid \alpha. \end{cases}$$

Lemma 4. Let $p \geq 3$ be a prime, and $\alpha_1, \dots, \alpha_n$ be positive integers, where $r = \alpha_1 + \cdots + \alpha_n \leq p-3$. We have

$$\sum_{\substack{1 \leq l_1, \dots, l_n \leq 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \cdots l_n^{\alpha_n}} \equiv \begin{cases} (-1)^n (n-1)! \frac{2r(r+1)}{r+2} B_{p-r-2} p^2 \pmod{p^3} & \text{if } 2 \nmid r; \\ (-1)^{n-1} (n-1)! \frac{2r}{r+1} B_{p-r-1} p \pmod{p^2} & \text{if } 2 \mid r. \end{cases}$$

Proof. For any positive integer $\alpha \leq p-3$, we have

$$\begin{aligned}
\sum_{p < l < 2p} \frac{1}{l^\alpha} &= \sum_{0 < l < p} \frac{1}{(l+p)^\alpha} \equiv \sum_{0 < l < p} \frac{(l-p)^\alpha}{(l^2-p^2)^\alpha} \\
&\equiv \sum_{0 < l < p} \frac{l^\alpha - \alpha p l^{\alpha-1} + \frac{\alpha(\alpha-1)}{2} p^2 l^{\alpha-2}}{l^{2\alpha} - \alpha p^2 l^{2(\alpha-1)}} \\
&\equiv \sum_{0 < l < p} \frac{(l^{2\alpha} + \alpha p^2 l^{2(\alpha-1)})(l^\alpha - \alpha p l^{\alpha-1} + \frac{\alpha(\alpha-1)}{2} p^2 l^{\alpha-2})}{l^{4\alpha}} \\
&\equiv \sum_{0 < l < p} \frac{1}{l^\alpha} - \alpha p \sum_{0 < l < p} \frac{1}{l^{\alpha+1}} + \frac{\alpha(\alpha+1)}{2} p^2 \sum_{0 < l < p} \frac{1}{l^{\alpha+2}} \pmod{p^3}.
\end{aligned}$$

Because $\sum_{0 < l < p} \frac{1}{l^{\alpha+2}} \equiv 0 \pmod{p}$, we have

$$\sum_{\substack{0 < l < 2p \\ l \neq p}} \frac{1}{l^\alpha} = \sum_{1 \leq l < p} \frac{1}{l^\alpha} + \sum_{p < l < 2p} \frac{1}{l^\alpha} \equiv 2 \sum_{0 < l < p} \frac{1}{l^\alpha} - \alpha p \sum_{0 < l < p} \frac{1}{l^{\alpha+1}} \pmod{p^3}.$$

By Lemma 3, we obtain

$$\sum_{\substack{0 < l < 2p \\ l \neq p}} \frac{1}{l^\alpha} \equiv \begin{cases} -\frac{2\alpha(\alpha+1)}{\alpha+2} p^2 B_{p-\alpha-2} \pmod{p^3} & \text{if } 2 \nmid \alpha; \\ \frac{2\alpha}{\alpha+1} p B_{p-\alpha-1} \pmod{p^2} & \text{if } 2 \mid \alpha. \end{cases}$$

This proves the lemma for $n=1$. Now assume the lemma is true when the number of variables is less than n . We have

$$\begin{aligned}
\sum_{\substack{1 \leq l_1, \dots, l_n < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} \dots l_n^{\alpha_n}} &= \sum_{\substack{1 \leq l_1, \dots, l_{n-1} < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} \dots l_{n-1}^{\alpha_{n-1}}} \left(\sum_{\substack{1 \leq l_n < 2p \\ l_n \neq p}} \frac{1}{l_n^{\alpha_n}} - \sum_{i=1}^{n-1} \frac{1}{l_i^{\alpha_n}} \right) \\
&= \left(\sum_{\substack{1 \leq l_1, \dots, l_{n-1} < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} \dots l_{n-1}^{\alpha_{n-1}}} \right) \left(\sum_{\substack{1 \leq l_n < 2p \\ l_n \neq p}} \frac{1}{l_n^{\alpha_n}} \right) \\
&\quad - \sum_{\substack{1 \leq l_1, \dots, l_{n-1} < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1 + \alpha_n} \dots l_{n-1}^{\alpha_{n-1}}} - \dots \\
&\quad - \sum_{\substack{1 \leq l_1, \dots, l_{n-1} < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} \dots l_{n-1}^{\alpha_{n-1} + \alpha_n}}.
\end{aligned}$$

From the assumption, we have

$$\left(\sum_{\substack{1 \leq l_1, \dots, l_{n-1} < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} \dots l_{n-1}^{\alpha_{n-1}}} \right) \left(\sum_{\substack{1 \leq l_n < 2p \\ l_n \neq p}} \frac{1}{l_n^{\alpha_n}} \right) \equiv \begin{cases} 0 \pmod{p^3} & \text{if } 2 \nmid r; \\ 0 \pmod{p^2} & \text{if } 2 \mid r. \end{cases}$$

If r is odd, then

$$\begin{aligned}
\sum_{\substack{1 \leq l_1, \dots, l_n < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_n^{\alpha_n}} &\equiv -(n-1) \sum_{\substack{1 \leq l_1, \dots, l_{n-1} < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1 + \alpha_n} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1}}} \\
&\equiv -(n-1)(-1)^{n-1}(n-2)! \frac{2r(r+1)}{r+2} B_{p-r-2} p^2 \\
&\equiv (-1)^n (n-1)! \frac{2r(r+1)}{r+2} B_{p-r-2} p^2 \pmod{p^3}.
\end{aligned}$$

If r is even, similarly we can derive

$$\sum_{\substack{1 \leq l_1, \dots, l_n < 2p \\ l_i \neq l_j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_n^{\alpha_n}} \equiv (-1)^{n-1} (n-1)! \frac{2r}{r+1} B_{p-r-1} p \pmod{p^2}.$$

The proof of Lemma 4 is complete by induction on n . \square

By letting $r = n = 1$ in this lemma, we obtain the following corollary.

Corollary 2. *Let α be a positive integer and $p \geq \alpha + 3$ be a prime. Then*

$$\sum_{1 \leq l < 2p, l \neq p} \frac{1}{l^\alpha} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } 2 \nmid \alpha; \\ 0 \pmod{p} & \text{if } 2 \mid \alpha. \end{cases}$$

Lemma 5. *Let $p > 5$ be a prime. We have*

$$\sum_{\substack{l_1 + \dots + l_5 = 2p \\ 1 \leq l_1, \dots, l_5 < p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv 2 \cdot 4! B_{p-5} \pmod{p}.$$

Proof. Let $u_i = l_1 + \dots + l_i, 1 \leq i \leq 4$. We have

$$\begin{aligned}
\sum_{\substack{l_1 + \dots + l_5 = 2p \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} &= \frac{1}{2p} \sum_{\substack{l_1 + \dots + l_5 = 2p \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{l_1 + l_2 + \dots + l_5}{l_1 l_2 \dots l_5} \\
&= \frac{5}{2p} \sum_{\substack{l_1 + \dots + l_4 < 2p \\ l_1, \dots, l_4, u_4 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4} \\
&= \frac{5 \cdot 4}{2p} \sum_{\substack{l_1, l_2, l_3 < u_4 < 2p \\ l_1, l_2, l_3, u_4, u_4 - u_3 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 u_4} = \dots \\
&= \frac{5!}{2p} \sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2 - u_1, \dots, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4}.
\end{aligned} \tag{2.1}$$

Moreover,

$$\begin{aligned}
\sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2 - u_1, u_3 - u_2, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} &= \sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2, u_3, u_4 \in \mathcal{P}_p \\ u_2 - u_1, u_3 - u_2, u_4 - u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} \\
&+ \sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p, u_2 = p \\ u_1, u_2 - u_1, u_3 - u_2, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} \\
&+ \sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p, u_3 = p \\ u_1, u_2 - u_1, u_3 - u_2, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4}.
\end{aligned} \tag{2.2}$$

It is not hard to see

$$\begin{aligned}
\sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p, u_2 = p \\ u_1, u_2 - u_1, u_3 - u_2, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} &= \frac{1}{p} \sum_{1 \leq u_1 < p} \frac{1}{u_1} \sum_{\substack{p < u_3 < u_4 < 2p \\ u_3, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_3 u_4} \\
&= \frac{1}{2p} \left(\sum_{1 \leq u_1 < p} \frac{1}{u_1} \right) \left(\left(\sum_{p < u_3 < 2p} \frac{1}{u_3} \right)^2 - \sum_{p < u_3 < 2p} \frac{1}{u_3^2} \right) \\
&\equiv 0 \pmod{p^2}.
\end{aligned}$$

Here in the last congruence we use Corollary 1 and Corollary 2.

In the same way, we have

$$\begin{aligned}
\sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p, u_3 = p \\ u_1, u_2 - u_1, u_3 - u_2, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} &= \frac{1}{p} \sum_{\substack{1 \leq u_1 < u_2 < p \\ u_1, u_2 - u_1, u_2 \in \mathcal{P}_p}} \frac{1}{u_1 u_2} \sum_{p < u_4 < 2p} \frac{1}{u_4} \\
&= \frac{1}{p} \left(\left(\sum_{1 \leq u_1 < p} \frac{1}{u_1} \right)^2 - \sum_{1 \leq u_1 < p} \frac{1}{u_1^2} \right) \left(\sum_{p < u_4 < 2p} \frac{1}{u_4} \right) \\
&\equiv 0 \pmod{p^2}.
\end{aligned}$$

Hence from (2.2) we deduce that

$$\begin{aligned}
\sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2 - u_1, u_3 - u_2, u_4 - u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} &\equiv \sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2, u_3, u_4 \in \mathcal{P}_p \\ u_2 - u_1, u_3 - u_2, u_4 - u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} \\
&\equiv \sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} - T_1 - T_2 - T_3 \pmod{p^2},
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
T_1 &= \sum_{\substack{u_1 < u_1 + p < u_3 < u_4 < 2p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1(u_1 + p)u_3 u_4}, \quad T_2 = \sum_{\substack{u_1 < u_2 < u_2 + p < u_4 < 2p \\ u_1, u_2, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2(u_2 + p)u_4}, \\
T_3 &= \sum_{\substack{u_1 < u_2 < u_3 < u_3 + p < 2p \\ u_1, u_2, u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3(u_3 + p)}.
\end{aligned}$$

We have

$$\begin{aligned}
T_1 &= \sum_{\substack{u_1 < u_1+p < u_3 < u_4 < 2p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1(u_1+p)u_3u_4} \quad (\text{replace } u_3 \text{ by } u_3+p \text{ and } u_4 \text{ by } u_4+p) \\
&= \sum_{\substack{u_1 < u_3 < u_4 < p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1(u_1+p)(u_3+p)(u_4+p)} \\
&\equiv \sum_{\substack{u_1 < u_3 < u_4 < p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{u_1u_3u_4 - p(u_1u_3 + u_3u_4 + u_4u_1)}{u_1u_1^2u_3^2u_4^2} \\
&\equiv \sum_{\substack{u_1 < u_3 < u_4 < p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1^2u_3u_4} - p \sum_{\substack{u_1 < u_3 < u_4 < p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1^2u_3u_4^2} - p \sum_{\substack{u_1 < u_3 < u_4 < p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1^3u_3u_4} \\
&\quad - p \sum_{\substack{u_1 < u_3 < u_4 < p \\ u_1, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1^2u_3^2u_4} \\
&\equiv \sum_{\substack{u_1 < u_2 < u_3 < p \\ u_1, u_2, u_3 \in \mathcal{P}_p}} \frac{1}{u_1^2u_2u_3} - p \sum_{\substack{u_1 < u_2 < u_3 < p \\ u_1, u_2, u_3 \in \mathcal{P}_p}} \frac{1}{u_1^2u_2u_3^2} - p \sum_{\substack{u_1 < u_2 < u_3 < p \\ u_1, u_2, u_3 \in \mathcal{P}_p}} \frac{1}{u_1^3u_2u_3} \\
&\quad - p \sum_{\substack{u_1 < u_2 < u_3 < p \\ u_1, u_2, u_3 \in \mathcal{P}_p}} \frac{1}{u_1^2u_2^2u_3} \pmod{p^2}.
\end{aligned}$$

Here in the last congruence we just replace the variables u_3 by u_2 and u_4 by u_3 .

Similarly,

$$\begin{aligned}
T_2 &= \sum_{u_1 < u_2 < u_2+p < u_4 < 2p} \frac{1}{u_1u_2(u_2+p)u_4} \quad (\text{replace } u_4 \text{ by } u_3+p) \\
&= \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2(u_2+p)(u_3+p)} \\
&\equiv \sum_{u_1 < u_2 < u_3 < p} \frac{u_2u_3 - p(u_2+u_3)}{u_1u_2u_2^2u_3^2} \\
&\equiv \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2^2u_3} - p \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2^2u_3^2} - p \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2^3u_3} \pmod{p^2}
\end{aligned}$$

and

$$\begin{aligned}
T_3 &= \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2u_3(u_3+p)} \\
&\equiv \sum_{u_1 < u_2 < u_3 < p} \frac{u_3 - p}{u_1u_2u_3^3} \\
&\equiv \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2u_3^2} - p \sum_{u_1 < u_2 < u_3 < p} \frac{1}{u_1u_2u_3^3} \pmod{p^2}.
\end{aligned}$$

Hence by Lemma 3, we have

$$\begin{aligned}
T_1 + T_2 + T_3 &= \sum_{u_1 < u_2 < u_3 < p} \left(\frac{1}{u_1^2 u_2 u_3} + \frac{1}{u_1 u_2^2 u_3} + \frac{1}{u_1 u_2 u_3^2} \right) \\
&\quad - p \sum_{u_1 < u_2 < u_3 < p} \left(\frac{1}{u_1^2 u_2 u_3^2} + \frac{1}{u_1^2 u_2^2 u_3} + \frac{1}{u_1 u_2^2 u_3^2} \right) \\
&\quad - p \sum_{u_1 < u_2 < u_3 < p} \left(\frac{1}{u_1^3 u_2 u_3} + \frac{1}{u_1 u_2^3 u_3} + \frac{1}{u_1 u_2 u_3^3} \right) \\
&\equiv \frac{1}{2} \left(\sum_{\substack{u_1, u_2, u_3 < p \\ u_i \neq u_j}} \frac{1}{u_1^2 u_2 u_3} - p \sum_{\substack{u_1, u_2, u_3 < p \\ u_i \neq u_j}} \frac{1}{u_1^2 u_2 u_3^2} - p \sum_{\substack{u_1, u_2, u_3 < p \\ u_i \neq u_j}} \frac{1}{u_1^3 u_2 u_3} \right) \\
&\equiv \frac{4}{5} p B_{p-5} \pmod{p^2}.
\end{aligned}$$

Substituting this congruence into (2.3) and combining with (2.1), we obtain

$$\sum_{\substack{l_1 + \dots + l_5 = 2p \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv \frac{5!}{2p} \left(\sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} - \frac{4}{5} p B_{p-5} \right) \pmod{p}.$$

By Lemma 4, we have

$$\sum_{\substack{1 \leq u_1 < \dots < u_4 < 2p \\ u_1, u_2, u_3, u_4 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} = \frac{1}{4!} \sum_{\substack{1 \leq u_1, \dots, u_4 < 2p \\ u_i \neq u_j, u_i \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} \equiv -\frac{2}{5} p B_{p-5} \pmod{p^2}.$$

Hence

$$\sum_{\substack{l_1 + \dots + l_5 = 2p \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} \equiv -3 \cdot 4! B_{p-5} \pmod{p}.$$

We observe that

$$\begin{aligned}
\sum_{\substack{l_1 + l_2 + \dots + l_5 = 2p \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} &= \sum_{\substack{l_1 + l_2 + \dots + l_5 = 2p \\ l_1, \dots, l_5 < p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} + 5 \sum_{\substack{l_1 + l_2 + \dots + l_5 = p \\ l_1, \dots, l_5 < p}} \frac{1}{(l_1 + p) l_2 l_3 l_4 l_5} \\
&\equiv S_5^{(2)}(p) + 5 S_5^{(1)}(p) \pmod{p}.
\end{aligned}$$

By (1.2), we have $S_5^{(1)}(p) \equiv -4! B_{p-5} \pmod{p}$. Hence we deduce that

$$S_5^{(2)}(p) \equiv \sum_{\substack{l_1 + l_2 + \dots + l_5 = 2p \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} - 5 S_5^{(1)}(p) \equiv 2 \cdot 4! B_{p-5} \pmod{p}.$$

This completes the proof. \square

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. For any 5-tuples (l_1, \dots, l_5) of integers satisfying $l_1 + \dots + l_5 = 2p^{r+1}$, $1 \leq l_i < p^{r+1}$, $l_i \in \mathcal{P}_p$, $1 \leq i \leq 5$, we rewrite them as

$$l_i = x_i p^r + y_i, \quad 0 \leq x_i < p, \quad 1 \leq y_i < p^r, \quad y_i \in \mathcal{P}_p, \quad 1 \leq i \leq 5.$$

Since

$$\left(\sum_{i=1}^5 x_i\right)p^r + \sum_{i=1}^5 y_i = 2p^{r+1},$$

we know there exists $1 \leq a \leq 4$ such that

$$\begin{cases} x_1 + \cdots + x_5 = 2p - a \\ y_1 + \cdots + y_5 = ap^r \end{cases}.$$

By (ii) of Lemma 2, we have $C_a \equiv 0 \pmod{p}$ for $1 \leq a \leq 4$. Hence

$$\begin{aligned} S_5^{(2)}(p^{r+1}) &= \sum_{\substack{l_1 + \cdots + l_5 = 2p^{r+1} \\ l_i \in \mathcal{P}_p, l_i < p^{r+1}}} \frac{1}{l_1 l_2 \cdots l_5} \\ &= \sum_{a=1}^4 \sum_{\substack{x_1 + \cdots + x_5 = 2p - a \\ 0 \leq x_i < p}} \sum_{\substack{y_1 + \cdots + y_5 = ap^r \\ y_i \in \mathcal{P}_p, y_i < p^r}} \frac{1}{(x_1 p^r + y_1) \cdots (x_5 p^r + y_5)} \\ &\equiv C_1 S_5^{(1)}(p^r) + C_2 S_5^{(2)}(p^r) + C_3 S_5^{(3)}(p^r) + C_4 S_5^{(4)}(p^r) \pmod{p^{r+1}}. \end{aligned}$$

By (i) of Lemma 1, we have $S_5^{(3)}(p^r) \equiv -S_5^{(2)}(p^r) \pmod{p^r}$ and $S_5^{(4)}(p^r) \equiv -S_5^{(1)}(p^r) \pmod{p^r}$. Hence

$$S_5^{(2)}(p^{r+1}) \equiv (C_1 - C_4)S_5^{(1)}(p^r) + (C_2 - C_3)S_5^{(2)}(p^r) \pmod{p^{r+1}}. \quad (3.1)$$

By (ii) of Lemma 1 we have

$$\begin{aligned} S_5^{(1)}(p^{r+1}) &\equiv \sum_{k=1}^4 S_5^{(k)}(p^r) \binom{p-k+4}{4} \\ &\equiv \left(\binom{p+3}{4} - \binom{p}{4} \right) S_5^{(1)}(p^r) + \left(\binom{p+2}{4} - \binom{p+1}{4} \right) S_5^{(2)}(p^r) \quad (3.2) \\ &= \frac{p(p^2+1)}{2} S_5^{(1)}(p^r) + \frac{p(p^2-1)}{6} S_5^{(2)}(p^r) \pmod{p^{r+1}}. \end{aligned}$$

From (3.1) and (3.2), by induction on r we can prove $S_5^{(1)}(p^r) \equiv S_5^{(2)}(p^r) \equiv 0 \pmod{p^{r-1}}$. Combining this with (3.1) and (3.2), by Lemma 2 we deduce that

$$\begin{aligned} S_5^{(1)}(p^{r+1}) &\equiv \frac{1}{2} p S_5^{(1)}(p^r) - \frac{1}{6} p S_5^{(2)}(p^r) \pmod{p^{r+1}}, \\ S_5^{(2)}(p^{r+1}) &\equiv -\frac{3}{2} p S_5^{(1)}(p^r) + \frac{1}{2} p S_5^{(2)}(p^r) \pmod{p^{r+1}}. \end{aligned} \quad (3.3)$$

From (3.3) we have

$$S_5^{(2)}(p^{r+1}) \equiv -3S_5^{(1)}(p^{r+1}) \pmod{p^{r+1}}, \quad r \geq 1. \quad (3.4)$$

Substituting this result into the first congruence of (3.3), we obtain

$$S_5^{(1)}(p^{r+1}) \equiv p S_5^{(1)}(p^r) \pmod{p^{r+1}}, \quad r \geq 2. \quad (3.5)$$

By (1.2), we have $S_5^{(1)}(p) \equiv -4!B_{p-5} \pmod{p}$. By Lemma 5 we have $S_5^{(2)}(p) \equiv 2 \cdot 4!B_{p-5} \pmod{p}$. Hence from (3.3) we deduce that

$$S_5^{(1)}(p^2) \equiv \frac{1}{2} p S_5^{(1)}(p) - \frac{1}{6} p S_5^{(2)}(p) \equiv -\frac{5!}{6} p B_{p-5} \pmod{p^2}.$$

Now using (3.5) and by induction on r we prove $S_5^{(1)}(p^r) \equiv -\frac{5!}{6}p^{r-1}B_{p-5} \pmod{p^r}$ for any prime $p > 5$ and integer $r \geq 2$. \square

Proof of Theorem 2. Let $n = mp^r$, where p does not divide m . For any 5-tuples (l_1, \dots, l_5) of integers satisfying $l_1 + \dots + l_5 = n$, $l_i \in \mathcal{P}_p$, $1 \leq i \leq 5$, we rewrite them as

$$l_i = x_i p^r + y_i, \quad x_i \geq 0, \quad 1 \leq y_i < p^r, \quad y_i \in \mathcal{P}_p, \quad 1 \leq i \leq 5.$$

Since

$$\left(\sum_{i=1}^5 x_i\right)p^r + \sum_{i=1}^5 y_i = mp^r,$$

we know there exists $1 \leq a \leq 4$ such that

$$\begin{cases} x_1 + \dots + x_5 = m - a \\ y_1 + \dots + y_5 = ap^r \end{cases}.$$

For $1 \leq a \leq 4$, the equation $x_1 + x_2 + x_3 + x_4 + x_5 = m - a$ has $\binom{m-a+4}{4}$ solutions $(x_1, x_2, x_3, x_4, x_5)$ of nonnegative integers. Hence

$$\begin{aligned} \sum_{\substack{l_1+l_2+\dots+l_5=n \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} &= \sum_{\substack{l_1+\dots+l_5=mp^r \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_5} \\ &= \sum_{a=1}^4 \sum_{\substack{x_1+\dots+x_5=m-a \\ 0 \leq x_i < p}} \sum_{\substack{y_1+\dots+y_5=ap^r \\ y_i \in \mathcal{P}_p, y_i < p^r}} \frac{1}{(x_1 p^r + y_1) \dots (x_5 p^r + y_5)} \\ &\equiv \binom{m+3}{4} S_5^{(1)}(p^r) + \binom{m+2}{4} S_5^{(2)}(p^r) \\ &\quad + \binom{m+1}{4} S_5^{(3)}(p^r) + \binom{m}{4} S_5^{(4)}(p^r) \pmod{p^r}. \end{aligned} \tag{3.6}$$

According to $r = 1$ or $r \geq 2$, we split our proof into two cases.

(i) If $r = 1$, then from (1.2) we know $S_5^{(1)}(p) \equiv -4!B_{p-5} \pmod{p}$. By Lemma 5, we deduce that $S_5^{(2)}(p) \equiv -2S_5^{(1)}(p) \pmod{p}$. By (i) of Lemma 1, we know $S_5^{(3)}(p) \equiv 2S_5^{(1)}(p) \pmod{p}$ and $S_5^{(4)}(p) \equiv -S_5^{(1)}(p) \pmod{p}$. Hence from (3.6) we have

$$\begin{aligned} \sum_{\substack{l_1+l_2+\dots+l_5=n \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} &\equiv \left(\binom{m+3}{4} - 2\binom{m+2}{4} + 2\binom{m+1}{4} - \binom{m}{4} \right) S_5^{(1)}(p) \\ &\equiv \frac{1}{6}(5m + m^3) S_5^{(1)}(p) \pmod{p}. \end{aligned}$$

Since $S_5^{(1)}(p) \equiv -4!B_{p-5} \pmod{p}$ and $m = \frac{n}{p}$, we complete the proof of (i).

(ii) If $r \geq 2$, then from (3.4) we deduce that for any integer $r \geq 2$, $S_5^{(2)}(p^r) \equiv -3S_5^{(1)}(p^r) \pmod{p^r}$. By (i) of Lemma 1, we obtain

$$S_5^{(3)}(p^r) \equiv -S_5^{(2)}(p^r) \equiv 3S_5^{(1)}(p^r) \pmod{p^r}, \quad S_5^{(4)}(p^r) \equiv -S_5^{(1)}(p^r) \pmod{p^r}.$$

Hence from (3.6) we obtain

$$\begin{aligned} \sum_{\substack{l_1+l_2+\dots+l_5=n \\ l_1, \dots, l_5 \in \mathcal{P}_p}} \frac{1}{l_1 l_2 l_3 l_4 l_5} &\equiv \left(\binom{m+3}{4} - 3 \binom{m+2}{4} + 3 \binom{m+1}{4} - \binom{m}{4} \right) S_5^{(1)}(p^r) \\ &\equiv m S_5^{(1)}(p^r) \pmod{p^r}. \end{aligned}$$

By Theorem 1, we have $S_5^{(1)}(p^r) \equiv -\frac{5!}{6} p^{r-1} B_{p-5} \pmod{p^r}$. This completes the proof of (ii). \square

As we mentioned earlier, naturally we can ask the following question: Can we find two arithmetical functions $a(n)$ and $b(n)$ such that

(i) for any odd integer $n \geq 7$ and prime $p > n$,

$$\sum_{\substack{l_1+l_2+\dots+l_n=p^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n} \equiv a(n) p^{r-1} B_{p-n} \pmod{p^r};$$

(ii) for any even integer $n \geq 6$ and prime $p > n$,

$$\sum_{\substack{l_1+l_2+\dots+l_n=p^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n} \equiv b(n) p^r B_{p-n-1} \pmod{p^{r+1}}.$$

At this stage, we are not able to answer this question. We believe such $a(n)$ and $b(n)$ exist but to solve this problem one may need to develop some new ideas or methods.

REFERENCES

- [1] C. Ji, A simple proof of a curious congruence by Zhao, Proc. Amer. Math. Soc. 133 (2005), 3469–3472.
- [2] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, Ann. Math. 39 (1938), 350–360.
- [3] Z. Sun, Congruence concerning Bernoulli numbers and Bernoulli polynomials, Discrete Appl. Math 105 (2000), 193–223.
- [4] L. Wang and T. Cai, A curious congruence modulo prime powers, J. Number Theory 144 (2014), 15–24.
- [5] B. Xia and T. Cai, Bernoulli numbers and congruence for harmonic sums, Int. J. Number Theory 06 (2010), 849–855.
- [6] J. Zhao, Bernoulli numbers, Wolstenholme’s theorem, and p^5 variations of Lucas’ theorem, J. Number Theory 123 (2007), 18–26.
- [7] J. Zhao, A super congruence involving multiple harmonic sums, preprint in Arxiv, <http://arxiv.org/abs/1404.3549>
- [8] X. Zhou and T. Cai, A generalization of a curious congruence on harmonic sums, Proc. Amer. Math. Soc. 135 (2007), 1329–1333.

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE, 119076, SINGAPORE

E-mail address: wangliuquan@nus.edu.sg; mathlqwang@163.com